Parametric Discrete Morse Theory

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Contents

Introduction

1. Why parametric Morse theory?
2. Developped by H. King, K. Knudson and N. Mramor.
3. Analogy with the smooth setting.

Discrete Morse Theory

Definition (Discrete Morse function)

Let $f : K \to \mathbb{R}$ be a real-valued function. If conditions (1) and (2) below are satisfied, we say f is a discrete Morse function on K.

(1) $\sum_{\beta \in \mathcal{B}_\alpha^F} \mathbb{1}_{\{f(\beta) \geq f(\alpha)\}} \leq 1, \ \forall \alpha \in K$

(2) $\sum_{\beta \in \mathcal{B}_\alpha^C} \mathbb{1}_{\{f(\beta) \leq f(\alpha)\}} \leq 1, \ \forall \alpha \in K$

If both sums are zero for the same given simplex $\alpha \in K$, then we say that $\alpha \in K$ is a **critical simplex**.

Discrete Morse Theory

Let $f : K \to \mathbb{R}$ be a discrete Morse function on K.

Definition (Discrete gradient vector field)

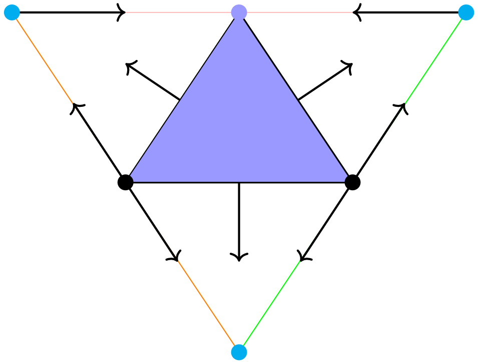
The ***discrete gradient vector field of*** f *is the set of all pairs* $\{\alpha^{(p)} < \beta^{(p+1)}\}$ *with* $f(\alpha) \geq f(\beta)$.

Let V be the discrete gradient vector field of f.

Definition (Gradient path)

A ***gradient path of*** f *is a family* $\{\alpha_1^{(p)}, \beta_1^{(p+1)}, \dots, \alpha_n^{(p)}, \beta_n^{(p+1)}, \alpha_{n+1}^{(p)}\}$ *with* $(\alpha_i, \beta_i) \in V$, $\alpha_i < \beta_i > \alpha_{i+1}$ *and* $\alpha_i \neq \alpha_{i+1}$ *for all* $i \in \{1, \dots, n\}$.

Discrete Morse Theory



Parametric Morse Theory

Context.

1. A simplicial complex K,
2. a sequence $\{(f_i, V_i)\}_{i=1}^n$,

where each function $f_i : K \to \mathbb{R}$ is a discrete Morse function on K with associated gradient vector field V_i.

Idea. Tracking critical cells along $\{(f_i, V_i)\}_{i=1}^n$.

Parametric Morse Theory

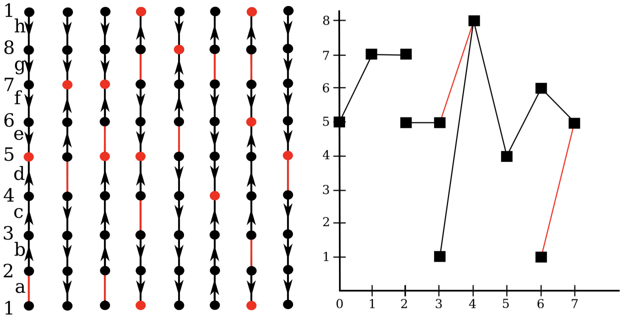
Definition (Forward and strong connections)

1. *k-cells α_i and α_j, with $i \neq j$*
2. *critical for f_i and f_j respectively*

We say there is a ***forward connection*** *from α_i to α_j, if there is a k-cell γ with a V_i-path $\{\alpha_i, \ldots, \gamma\} \subseteq K^{k-1} \cup K^k$ and a V_j-path $\{\gamma, \ldots, \alpha_j\} \subseteq K^k \cup K^{k+1}$. If there is a forward connection from α_i to α_j and vice-versa, then there is a* ***strong connection*** *between α_i and α_j.*

Parametric Morse Theory

Finite Morse parametrization and birth-death diagram.



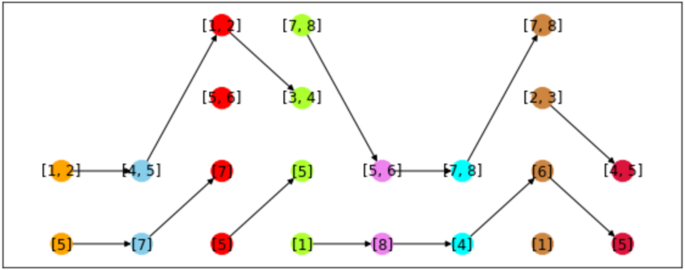
Parametric Morse Theory

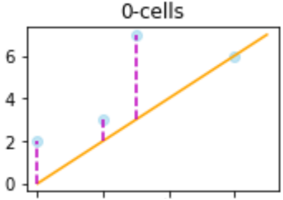
Definition (Birth and death)

We say α is born (resp. dies) at index i if no strong connection exists between α and any cell that is critical for V_{i-1} (resp. V_{i+1}). If there is cell $\beta \in K$ that is critical for V_{i+1}, such that the only strong connection of α and β is between themselves, then we say α moves (or mutes) to β.

Parametric Morse Theory

Birth-death diagram and parametric persistence diagram.





A note on Building Discrete Morse Functions

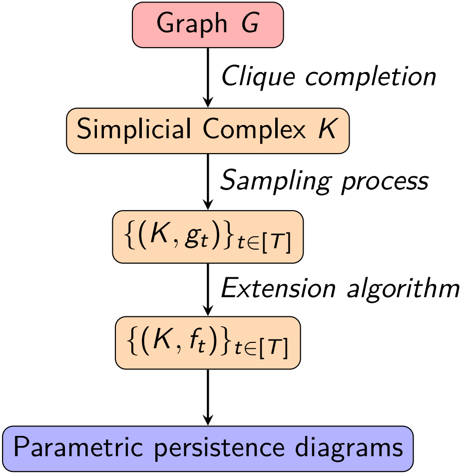
1. Real datasets with node values
2. Need to generate Morse functions

We use a simple algorithm to extend a node function $g : G = K^{(0)} \to \mathbb{R}$ on a simplicial complex K to a discrete Morse function $f : K \to \mathbb{R}$. The algorithm is designed with complexity $O(m)$, where m is the number of simplices in K.

Remark

We consider examples where the initial node function $g : G = K^{(0)} \to \mathbb{R}$ is **injective**. This way, the algorithm produces significantly less critical cells.

Method and pipeline



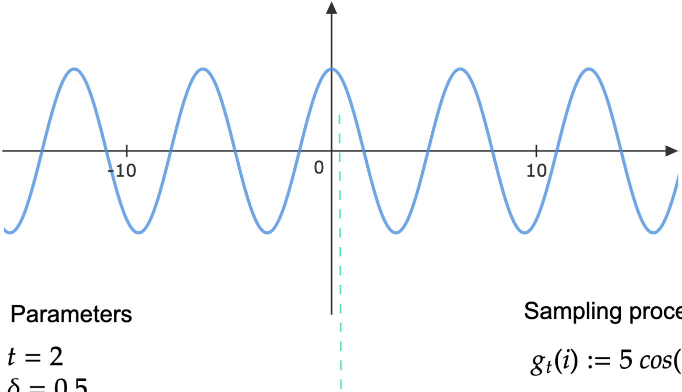
Sampling Process

1. Input mapping $\mathbf{h} : G \to \mathcal{C}^0(\mathbb{R})$
2. Input parameters $(x_0, \delta, T) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{N}$
3. Assigning node values at time $t \in [T]$ with

$$g_t : \begin{cases} G \to \mathbb{R} \\ i \to g_t(i) := h_i(x_0 + t\delta) \end{cases}$$

Sampling Process

Example : sampling node values on a cosine function



Parameters

$t = 2$

$\delta = 0.5$

$x_0 = 0$

$h_i : x \mapsto 5\, cos(x)$

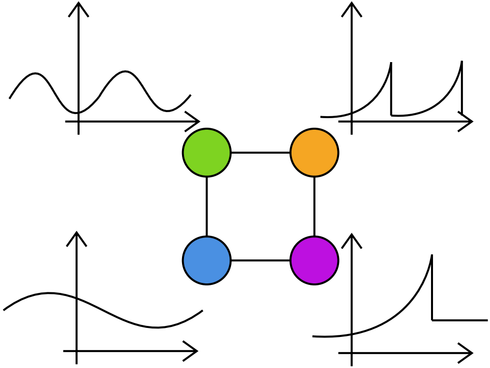
$x = x_0 + t\delta = 1$

Sampling process

$g_t(i) := 5\, cos(1)$

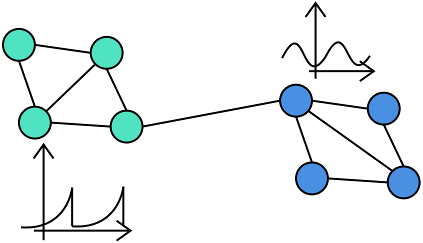
Input example

Firing models amongst populations of neurons.

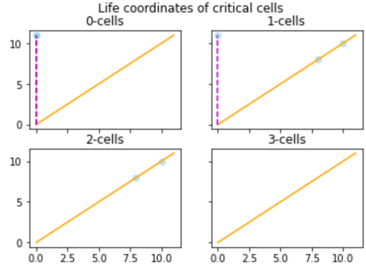


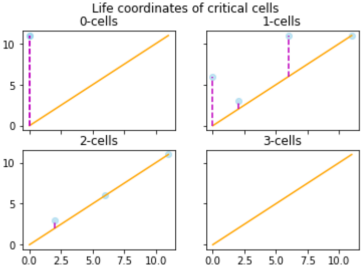
Results : Parametric Persistence Diagrams

Stochastic block model.

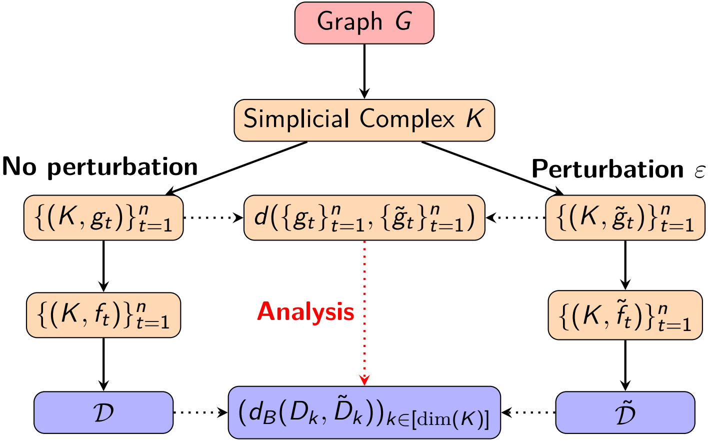


Results : Parametric Persistence Diagrams



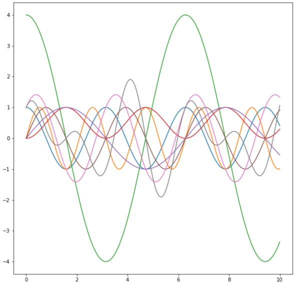


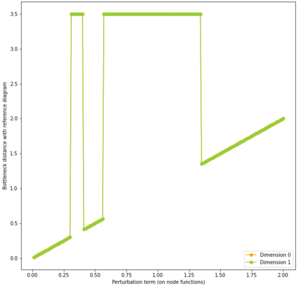
Stability Analysis



Results : Stability Curves

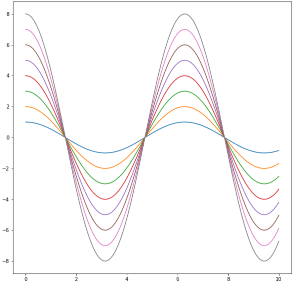
Different periodicity in sampling node functions.

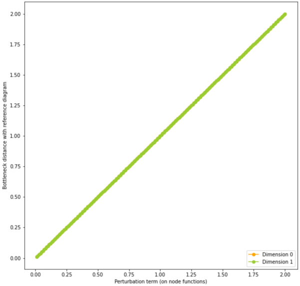




Results : Stability Curves

Same periodicity in sampling node functions.





Interpretation

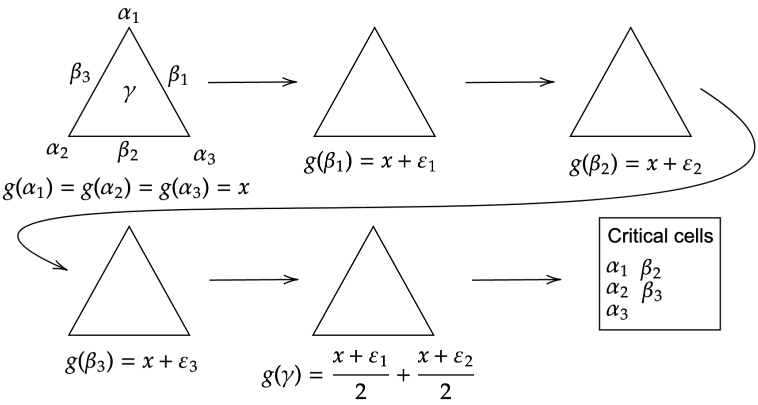
1. Same periodicity yields constant Bottleneck distances (linear curve).
2. Indicates that changes happen when both an increase and a decrease in vertex values happen at the same time.
3. For Lipschitz functions, a perturbation $\varepsilon \to 0$ leads to no change in the output except at local maxima and minima.

Future Work

1. Extension algorithms
2. Interpreting complex examples
3. Real-world datasets
4. Discontinuous functions - identify time of firing.

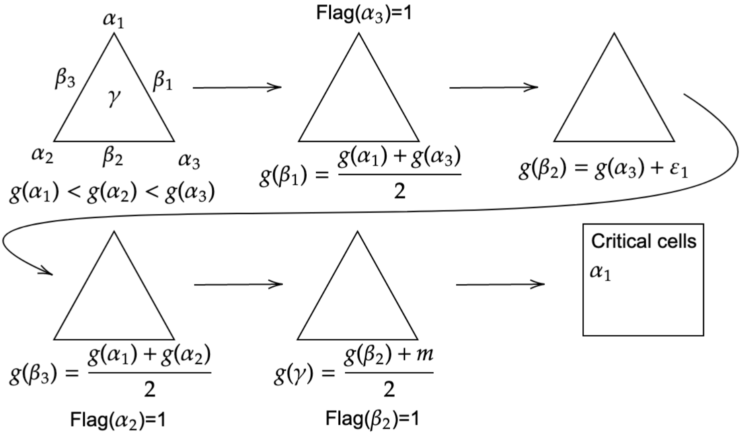
Thank you for listening!

Bonus



Assuming that $\varepsilon_3 < \varepsilon_2 < \varepsilon_1$

Bonus



With $m = max(g(\beta_1), g(\beta_3))$

Bonus : the stochastic block model