

Polynomials

Bridging course in mathematics

Lesson 1



- 1 Sets of numbers
- 2 Polynomials
- 3 Polynomial operations
- 4 Factorization

$\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$

We shall typically work with **real** numbers, whose set is denoted by $\mathbb{R}$

The set of real numbers contains important subsets:

- **natural** numbers, denoted by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- **integer** numbers, denoted by $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- **rational** numbers, denoted by $\mathbb{Q}$:
a rational number can be written as ratio m/n of two integers,
with $n \neq 0$

Definition

- A **real polynomial in** the variable **x** is an algebraic expression of the form

$$A_n(x) = a_0 + a_1x + \dots + a_nx^n,$$

where $a_0, a_1, \dots, a_n$ are real numbers (called the **coefficients** of the polynomial) and $a_n \neq 0$

- The summands are said **monomials**
- The **degree** of a polynomial is the maximum power among the non-zero monomials appearing in the expression

Property

Two polynomials are equal if they have the same degree and the coefficients of the monomials with equal degree coincide

Adding polynomials

The **sum** of two polynomials is the polynomial obtained by adding the coefficients of the monomials of same degree

$$A_n(x) + B_m(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_m + b_m)x^m$$

Example

$$\begin{aligned} & (x^2 + 2x - 5) + (x^3 - x + 2) = \\ & (0x^3 + x^2 + 2x - 5) + (x^3 + 0x^2 - x + 2) = \\ & (0 + 1)x^3 + (1 + 0)x^2 + (2 - 1)x + (-5 + 2) = \\ & \quad x^3 + x^2 + x - 3 \end{aligned}$$

Multiplying polynomials

The **product** is the polynomial of the form

$$A_n(x) \cdot B_m(x) = C_{n+m}(x) = c_0 + c_1x + \dots + c_{n+m}x^{n+m},$$

whose coefficients are given by:

$$\begin{aligned} c_0 &= a_0b_0, & c_1 &= a_1b_0 + a_0b_1, \\ c_k &= a_0b_k + a_1b_{k-1} + a_2b_{k-2} + \dots + a_{k-1}b_1 + a_kb_0 \end{aligned}$$

In practice

$$\begin{aligned} (x-1)(x^2+x+1) &= \\ x^2(x-1) + x(x-1) + (x-1) &= \\ x^3 - x^2 + x^2 - x + x - 1 &= \\ x^3 - 1 \end{aligned}$$

Dividing polynomials

Given polynomials $A_n(x)$, $B_m(x)$ of degrees $n \geq m$, there exist polynomials $Q(x)$ and $R(x)$, called **quotient** and **remainder**, such that:

- the degree of $R(x)$ is less than m ;
- $A_n(x) = B_m(x) Q(x) + R(x)$

Definition

If $R(x) = 0$, one says that $B_m(x)$ divides $A_n(x)$

Therefore

The ratio of $A_n(x)$ and $B_m(x)$ can always be written as

$$\frac{A_n(x)}{B_m(x)} = Q(x) + \frac{R(x)}{B_m(x)}$$

where $\deg R < \deg B_m$

The method for dividing two polynomials consists in
dividing monomials by decreasing degree

Example

Let's divide

$$A(x) = 2x^4 + x^3 - x + 2$$

by

$$B(x) = x^2 + 3$$

$$R(x)$$





Example



$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2	$+3$
$2x^4$		$+6x^2$			$2x^2$	$+x - 6$
	$+x^3$	$-6x^2$	$-x$	$+2$	$\uparrow$	
	$+x^3$		$+3x$		$Q(x)$	
		$-6x^2$	$-4x$	$+2$		
		$-6x^2$		-18		
			$-4x$	$+20$	$\leftarrow$	$R(x)$

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)} = 2x^2 + x - 6 + \frac{-4x + 20}{x^2 + 3}$$

Factorizing a polynomial means transforming it into a **product**

Example

$$x^2 - 5x + 6 = (x - 3)(x - 2)$$

A polynomial that cannot be factorized is called **irreducible**;

Real polynomials of degree 1 are irreducible, and also those of degree 2 with negative discriminant.

Useful methods

Finding an overall common factor

$$x^4 - 3x^3 + 5x^2 = x^2(x^2 - 3x + 5)$$

Finding a partial factor

$$\begin{aligned} x^4 + a^2x^2 + b^2x^2 + a^2b^2 &= x^2(x^2 + a^2) + b^2(x^2 + a^2) = \\ &= (x^2 + a^2)(x^2 + b^2) \end{aligned}$$

Factor / remainder theorem

$(x - c)$ divides the polynomial $A_n(x)$ if and only if $A_n(c) = 0$

Consequences

- The binomial $x^n - a^n$ is always divisible by $x - a$;
if n is even, $x^n - a^n$ is also divisible by $x + a$
- The binomial $x^n + a^n$ is divisible by $x + a$ if n is odd;
if n is even, $x^n + a^n$ is neither divisible by $x + a$ nor by $x - a$

Remark

For polynomials with **integer coefficients**:

- the possible integer roots should be searched for among the factors with sign of the constant term, including 1;
- the possible rational roots should be looked for among the rationals of the form $\pm p/q$, where p divides the constant term, and q divides the leading coefficient